Journal of Computational and Applied Mathematics 236 (2011) 924–936

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving parametric piecewise polynomial systems

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ARTICLE INFO

MSC:
14M15
14Q10
41A15
41A46
65D07
65D10

Keywords:
Piecewise polynomial
Parametric piecewise polynomial system
Parametric semi-algebraic systems
Discriminant variety
Number of real zeros

ABSTRACT

We deal with C^r smooth continuity conditions for piecewise polynomial functions on Δ, where Δ is an algebraic hypersurface partition of a domain Ω in $\mathbb{R}^n$. Piecewise polynomial functions of degree, at most, k on Δ that are continuously differentiable of order r form a spline space C_k^r.

We present a method for solving parametric systems of piecewise polynomial equations of the form $\mathcal{Z}(f_1,\ldots,f_n) = \{X \in \Omega \mid f_1(V,X) = 0,\ldots,f_n(V,X) = 0\}$, where $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta)$, and $f_\omega|_{\sigma_i} \in \mathbb{Q}[V][X]$ for each n-cell σ_i in Δ, $V = (u_1,u_2,\ldots,u_\tau)$ is the set of parameters and $X = (x_1,x_2,\ldots,x_n)$ is the set of variables; $\sigma_1,\sigma_2,\ldots,\sigma_m$ are all the n-dimensional cells in Δ and $\Omega = \bigcup_{i=1}^m \sigma_i$.

Based on the discriminant variety method presented by Lazard and Rouillier, we show that solving a parametric piecewise polynomial system $\mathcal{Z}(f_1,\ldots,f_n)$ is reduced to the computation of discriminant variety of $\mathcal{Z}$. The variety can then be used to solve the parametric piecewise polynomial system.

We also propose a general method to classify the parameters of $\mathcal{Z}(f_1,\ldots,f_n)$. This method allows us to say that if there exist an open set of the parameters' space where the system admits exactly a given number of distinct torsion-free real zeros in every n-cells in Δ.

1. Introduction

Let $\mathbb{R}$ be the real number field. An element $X = (x_1,\ldots,x_n) \in \mathbb{R}^n$ will be called a point. We denote by $\mathbb{R}[x_1,\ldots,x_n]$ ($\mathbb{R}[X]$ for short) the polynomial ring in n variables $x_1,\ldots,x_n$ over $\mathbb{R}$.

Denote by $P_k[X]$ the collection of all these polynomials in $\mathbb{R}[X]$ with real coefficients and total degree $\le k$. A polynomial $p \in \mathbb{R}[X]$ is called an irreducible polynomial, if the polynomial cannot be exactly divided by any other polynomial except a constant or itself (in complex field $\mathbb{C}$). Algebraic hypersurface

$$\mathbf{V}(l_{ij}) : l(x_1,\ldots,x_n) = 0, \quad l(x_1,\ldots,x_n) \in P_k[X]$$

is called an irreducible algebraic hypersurface, if $l(x_1,\ldots,x_n)$ is an irreducible polynomial.

Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathbb{R}[X]$ (including the empty set). It is well known [1,2] that a basic closed semi-algebraic set $\sigma \subset \mathbb{R}^n$ can be described as

$$\sigma = \left\{ X \in \mathbb{R}^n \mid \bigwedge_{Q\in\mathcal{Q}} Q(X) = 0, \bigwedge_{P\in\mathcal{P}} P(X) \ge 0 \right\}. \tag{1}$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.05.008

The intersection of σ with algebraic variety $\mathbf{V}(g_1, \dots, g_k)$ is called a face of σ for any given $\{g_1, \dots, g_k\} \subseteq \mathscr{P}$. Obviously, a face of σ is a basic closed semi-algebraic set.

A simply connected basic closed semi-algebraic subset on an affine irreducible algebraic hypersurface is called an irreducible hypersurface patch in $\mathbb{R}^n$. Now we define a hereditary partition of a domain in $\mathbb{R}^n$.

Let Ω be an n-dimensional simply connected closed semi-algebraic domain in $\mathbb{R}^n$ (or subsets of $\mathbb{R}^n$). Δ is called a partition of the domain Ω and can be described as a finite collection of simply connected basic closed semi-algebraic subsets of $\mathbb{R}^n$ such that

(a) the faces of each element of Δ are elements of Δ, and the intersection of any two elements of Δ is an element of Δ;
(b) if $\delta \subset \tau$ with $\delta, \tau \in \Delta$, then $\dim(\delta) < \dim(\tau)$, and every maximal elements of Δ (with respect to inclusion) is an n-dimensional simply connected basic closed semi-algebraic subset of $\mathbb{R}^n$;
(c) the interior of each element of Δ is homeomorphic to an open i-cube $(0, 1)^i$, and its faces are algebraic hypersurface patches or semi-algebraic sets with dimensions $< n$;
(d) if two n-dimensional cells in Δ (we will sometimes refer to the k-dimensional elements of Δ as $k-$ *cells*) meet in a face of dimension $n-1$, then the face is an $(n-1)$-dimensional irreducible hypersurface patch in $\mathbb{R}^n$, and called an interior $(n-1)$-cell of Δ.

If two n-dimensional cells σ_i and σ_j in Δ intersect along an interior $(n-1)$-cell $\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$ contained in an affine irreducible algebraic hypersurface $\mathbf{V}(l_{ij})$, we say they are **adjacent**. Δ is said to be **hereditary** if for every $\tau \in \Delta$ (including the empty set), any two n-dimensional cells σ, σ' of Δ that contain τ can be connected by a sequence $\sigma = \sigma_1, \sigma_2, \dots, \sigma_m = \sigma'$ in Δ such that each σ_i is n-dimensional, each σ_i contains τ, and σ_i and σ_{i+1} are adjacent for each i.

Let Δ be a hereditary partition of a domain Ω in $\mathbb{R}^n$, let $\sigma_1, \sigma_2, \dots, \sigma_m$ be a given, fixed, ordering of the n-cells in Δ, and let $\Omega = \bigcup_{i=1}^m \sigma_i$. Now we give the definitions of $C^r(\Delta)$ and $C_k^r(\Delta)$ [3–8].

Definition 1.1. For a nonnegative integer r and a hereditary partition Δ of a domain Ω in $\mathbb{R}^n$, $C^r(\Delta)$ is the set of C^r functions f on Ω (that is, functions such that all rth order partial derivatives exist and are continuous on Ω) such that for every $\delta \in \Delta$ including those of dimension $< n$, the restriction $f|_\delta$ is a polynomial function $f|_\delta \in \mathbb{R}[x_1, \dots, x_n]$. $C_k^r(\Delta)$ is the subset of $f \in C^r(\Delta)$ such that the restriction of f to each cell in Δ is a polynomial function of degree k or less.

Remark 1. Since $\Omega = \bigcup_{i=1}^m \sigma_i$, we write the partition $\Delta = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$, and each element σ_i of Δ can be described as

$$\sigma_i = \left\{ x \in \mathbb{R}^n \mid g_1^{[i]}(x) \geq 0, \dots, g_{\eta_i}^{[i]}(x) \geq 0 \right\}, \tag{2}$$

where $g_1^{[i]}(x), \dots, g_{\eta_i}^{[i]}(x)$ are in $\mathbb{R}[x_1, \dots, x_n]$. The continuity of the function on Ω implies that only the cells $\sigma_1, \sigma_2, \dots, \sigma_m$ need to be considered and that differentiability follows from conditions on the intersections of the closure of these cells.

Let $P(\Delta) := \{f | f|_{\sigma_i} \in \mathbb{R}[x_1, \dots, x_n], i = 1, \dots, m\}$ be the set of piecewise polynomials defined on Δ.

The hereditary partition Δ defined as above (we will sometimes call it algebraic hypersurfaces partition because boundaries (faces) of each cell in the Δ are algebraic hypersurface patches) is new and more general compared with the polyhedral subdivision in Ref. [4] where the boundaries of each cell are only linear. Moreover, different partitions lead to different piecewise polynomials.

The elements of $C^r(\Delta)$ are known as C^r piecewise polynomials or C^r-splines (see [4,6,5,7,8]).

Let $f_1 \in C_{k_1}^{r_1}(\Delta), f_2 \in C_{k_2}^{r_2}(\Delta), \dots, f_s \in C_{k_s}^{r_s}(\Delta)$. Define

$$\mathcal{Z}(f_1, \dots, f_s) := \{(x_1, \dots, x_n) \in \Omega \mid f_i(x_1, \dots, x_n) = 0, i = 1, \dots, s\}.$$

We call $\mathcal{Z}(f_1, \dots, f_s)$ a **piecewise polynomial system** (piecewise algebraic variety in [4–7]). A point of the piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_s)$, or a common zero of piecewise polynomials $f_1, \dots, f_s$, is called torsion-free if it lies in interior of some n-cell in Δ.

Let $\mathbb{Q}$ be the rational number field and $V = (v_1, v_2, \dots, v_\tau)$ parameters, we denote by $\mathbb{R}[V][X]$ (resp. $\mathbb{Q}[V][X]$) the polynomials in n variables $X = (x_1, \dots, x_n)$ with coefficients in $\mathbb{R}[V]$ (resp. $\mathbb{Q}[V]$). Obviously, $p \in \mathbb{R}[V][X]$ can be viewed as $p = p(V, X) \in \mathbb{R}[V, X]$.

Definition 1.2. Let $f = f(V, X)$ be a piecewise function on Ω and $f|_{\sigma_i} \in \mathbb{R}[V][X]$ for each n-cell σ_i in Δ. $f \in C_k^r(\Delta)$ if and only if f is a C^r piecewise polynomial with respect to n variables X with degree k.

Definition 1.3. Let $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta), \omega = 1, \dots, s$, and $f_\omega|_{\sigma_i} \in \mathbb{R}[V][X]$ for each n-cell σ_i in Δ, where $V = (v_1, v_2, \dots, v_\tau)$ are viewed as parameters. Then $\mathcal{Z}(f_1, \dots, f_s)$ is called a parametric piecewise polynomial system. The system is a zero-dimensional if and only if for almost all the parametrics V's values satisfying $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta)$ for all $\omega \in \{1, \dots, s\}$, these sets

$$\Upsilon_i = \{X \in \mathbb{C}^n | f_\omega|_{\sigma_i}(V, X) = 0, \dots, f_\omega|_{\sigma_i}(V, X) = 0\}, \quad i = 1, \dots, m$$

are generally zero-dimensional in $\mathbb{C}^n$.

Piecewise polynomial systems have many applications in various scientific–academic as well as industrial domains, such as CAD, CAM, CAE and Image processing (see [4,6]).

Many problems in both practice and theory, for example, the construction of explicit interpolation schemes for spline spaces on given partition, blending curves and surfaces and computer graphics (see [4,6,7]), can be reduced to problems of solving parametric piecewise polynomial systems. It is obvious that the parametric piecewise polynomial system is also a kind of generalization of the parametric semi-algebraic system. In the literature [4], based on the method proposed in [9,10] and the partial cylindrical algebraic decomposition method (PCAD for short) proposed in [11–13], authors proposed a method for computing the sup of the number of the torsion-free real zeros of zero-dimensional parametric piecewise algebraic variety on a polyhedral n-complex and its distributions in n-dimensional cells, and also give an algorithm to produce a necessary and sufficient condition for a given zero-dimensional parametric piecewise algebraic variety to have a given number of distinct torsion-free real zeros in every n-cell in the n-complex.

Using PACD, Yang and Xia et al. [9,10] give a practical algorithm for solving parametric semi-algebraic system, the algorithm DISCRVAR in [9] can be used for solving parametric semi-algebraic system. Using a discriminant variety of the basic constructible set, Lazard and Rouillier (see [14]) recently proposed a new framework for studying the basic constructible set and the basic semi-algebraic set.

To describe our results more precisely, we first define the notion which will be used throughout the paper.

Notation 1. Let us consider the zero-dimensional parametric piecewise polynomial system

$$\mathcal{Z}(f_1, \ldots, f_n) = \{X \in \Omega | f_1(V, X) = 0, \ldots, f_n(V, X) = 0\},$$

where $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$, and $f_\omega|_{\sigma_i} \in \mathbb{Q}[V][X]$ for each n-cell σ_i in Δ, $V = (u_1, u_2, \ldots, u_\tau)$ is the set of parameters and $X = (x_1, x_2, \ldots, x_n)$ the set of variables; $\sigma_1, \sigma_2, \ldots, \sigma_m$ are all the n-dimensional cells in Δ and $\Omega = \bigcup_{i=1}^{m} \sigma_i$.

The aims of this paper are to establish the basic theory on C^r piecewise polynomial functions and $C^{\overline{r}}$ piecewise polynomial functions with different smoothness degree $\overline{r} = (r_{i_1 j_1}, \ldots, r_{i_t j_t})$, and to present theory and methods for solving zero-dimensional parametric piecewise polynomial systems.

In Section 2, necessary and sufficient conditions for piecewise polynomial function f be in $C^r(\Delta)$ and $C^{\overline{r}}(\Delta)$ are discussed respectively, which shows that coefficients of the piecewise polynomial f satisfy a linear equations system. We also obtain an existence theorem on $C^{\overline{r}_\omega}$ piecewise polynomial.

In Section 3, We show that solving $\mathcal{Z}(f_1, \ldots, f_n)$ amounts to solve m parametric semi-algebraic systems $\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}$ with $U = (u_1, u_2, \ldots, u_d)$ as parameters. We also present an algorithm to produce $\{\mathrm{SAS}\overline{S_1}, \mathrm{SAS}\overline{S_2}, \ldots, \mathrm{SAS}\overline{S_m}\}$ and $\Pi_d(\Theta_{\mathrm{BQ}})$, where Θ_{BQ} is the set of real solutions of the system BQ that corresponds to smooth continuity conditions of piecewise polynomials $f_1, \ldots, f_n$ (see Section 3 for details), and Π_d denote canonical projection from parameter's space V onto parameter's space U.

In Section 4, Based on the discriminant variety of basic parametric constructible set presented in [14], we show that solving $\mathcal{Z}(f_1, \ldots, f_n)$ is reduced to the computation of m discriminant varieties $W_{\mathrm{SAS}\overline{S_i}}$ of $\mathrm{SAS}\overline{S_i}$, $i = 1, \ldots, m$. The variety $\bigcup_{i=1}^{m} W_{\mathrm{SAS}\overline{S_i}}$ can then be used to solve the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ as long as one can describe $(\mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}})) \setminus (\bigcup_{i=1}^{m} W_{\mathrm{SAS}\overline{S_i}})$. This can be done by using the critical points method [2,15–17] and the Collins PCAD [11–13] or an *open* cylindrical algebraic decomposition [18].

In Section 5, based on the classification method presented in [15], we also propose a classification method and its algorithm to answer to the question "Given a parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ and integers $N_1, \ldots, N_m$, does there exist an open set $\mathcal{O}$ in the parameters' space such that for all $p_0 \in \mathcal{O}$, the zero-dimensional non-parametric piecewise polynomial system $\mathcal{Z}_{p_0}(f_1, \ldots, f_n)$ obtained by specializing at point p_0 have exactly $N_1, \ldots, N_m$ distinct torsion-free real zeros in n-cells $\sigma_1, \ldots, \sigma_m$ respectively? If yes, give explicitly a point $a \in \mathcal{O}$". In Section 6, we present some experimental results of our algorithms.

2. C^r piecewise polynomial

In this section, we establish the basic theory on C^r piecewise polynomial functions by using similar Wang's methods in [6,8], and obtain necessary and sufficient conditions on piecewise polynomial function f be in $C^r(\Delta)$. The results show that coefficients of the piecewise polynomial f satisfy a linear equations system.

Lemma 2.1. *Let $p(x_1, \ldots, x_n) \in P_k[X]$, and $q(x_1, \ldots, x_n)$ be an irreducible polynomial with total degree m. If the dimension of the intersection of the hypersurface $\mathbf{V}(p(x_1, \ldots, x_n))$ with the hypersurface $\mathbf{V}(q(x_1, \ldots, x_n))$ in $\mathbb{R}^n$, denoted $\dim(\mathbf{V}(p(x_1, \ldots, x_n)) \cap \mathbf{V}(q(x_1, \ldots, x_n)))$, is equal to $n - 1$. Then $p(x_1, \ldots, x_n)$ is divisible by $q(x_1, \ldots, x_n)$. Namely, there is a $r(x_1, \ldots, x_n) \in P_{k-m}[X]$, such that $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) r(x_1, \ldots, x_n)$.*

Proof. Because $q(x_1, \ldots, x_n)$ is an irreducible polynomial, $\mathbf{V}(q(x_1, \ldots, x_n))$ is an $(n - 1)$-dimensional affine variety in $\mathbb{C}^n$ (see [19]). If $p(x_1, \ldots, x_n)$ is not divisible by $q(x_1, \ldots, x_n)$, then $\mathbf{V}(q(x_1, \ldots, x_n)) \not\subseteq \mathbf{V}(p(x_1, \ldots, x_n))$ in $\mathbb{C}^n$, this implies that $\dim(\mathbf{V}(p(x_1, \ldots, x_n)) \cap \mathbf{V}(q(x_1, \ldots, x_n))) = n - 2$ in $\mathbb{C}^n$ by [19]. But $\dim(\mathbf{V}(p(x_1, \ldots, x_n)) \cap \mathbf{V}(q(x_1, \ldots, x_n))) = n - 1$ in $\mathbb{R}^n$. Therefore, there is a $r(x_1, \ldots, x_n) \in P_{k-m}[X]$, such that $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) r(x_1, \ldots, x_n)$. This completes the proof. $\square$

Theorem 2.2. *Let Δ be a hereditary partition of an n-dimensional simply connected closed semi-algebraic domain Ω in $\mathbb{R}^n$. Let the representation of function $z = f(x_1, \ldots, x_n)$ on the two arbitrary adjacent n-cells σ_i and σ_j in Δ be*

$$z = f|_{\sigma_i}(x_1, \ldots, x_n) \quad \text{and} \quad z = f|_{\sigma_j}(x_1, \ldots, x_n),$$

where $f|_{\sigma_i}(x_1, \ldots, x_n), f|_{\sigma_j}(x_1, \ldots, x_n) \in P_k[X]$. Then $z = f(x_1, \ldots, x_n) \in C^r(\sigma_i \cup \sigma_j)$ if and only if there is a polynomial $q_{ij}(x_1, \ldots, x_n) \in P_{k-(r+1)d}[X]$, such that

$$f|_{\sigma_i}(x_1, \ldots, x_n) - f|_{\sigma_j}(x_1, \ldots, x_n) = [l_{ij}(x_1, \ldots, x_n)]^{r+1} q_{ij}(x_1, \ldots, x_n), \tag{3}$$

where σ_i and σ_j intersect along an interior $(n-1)$-cell $\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$ contained in the affine hypersurface $\mathbf{V}(l_{ij})$, where $l_{ij}(x_1, \ldots, x_n) \in P_d[X]$ is an irreducible polynomial with total degree d.

Proof. Let r be a given positive integer, $0 \le r \le kd^{-1} - 1$. According to the given condition, $f(x_1, \ldots, x_n)$ is continuous everywhere in σ_{ij}. Hence $\eta(x_1, \ldots, x_n) = f|_{\sigma_i}(x_1, \ldots, x_n) - f|_{\sigma_j}(x_1, \ldots, x_n)$, is equal to zero everywhere in σ_{ij}. Because $(n-1)$-cell σ_{ij} is contained in the affine irreducible hypersurface $\mathbf{V}(l_{ij})$, $\dim(\mathbf{V}(\eta(x_1, \ldots, x_n)) \cap \mathbf{V}(l(x_1, \ldots, x_n))) = n - 1$. By Lemma 2.1, there is a polynomial $q_1(x_1, \ldots, x_n) \in P_{k-d}[X]$, such that

$$\eta(x_1, \ldots, x_n) = f|_{\sigma_i}(x_1, \ldots, x_n) - f|_{\sigma_j}(x_1, \ldots, x_n) = l_{ij}(x_1, \ldots, x_n) q_1(x_1, \ldots, x_n). \tag{4}$$

Also according to the property that the partial derivative of the first order of $\eta(x_1, \ldots, x_n)$ being zero in σ_{ij}, we know that

$$\begin{cases} \dfrac{\partial q_1}{\partial x_1} l_{ij}(x_1, \ldots, x_n) + q_1(x_1, \ldots, x_n) \left.\dfrac{\partial l_{ij}}{\partial x_1}\right|_{\sigma_{ij}} = 0, \\ \cdots \\ \dfrac{\partial q_1}{\partial x_n} l_{ij}(x_1, \ldots, x_n) + q_1(x_1, \ldots, x_n) \left.\dfrac{\partial l_{ij}}{\partial x_n}\right|_{\sigma_{ij}} = 0. \end{cases} \tag{5}$$

Since $l_{ij}(x_1, \ldots, x_n)$ is irreducible, by two equations in (5), we know that $q_1(x_1, \ldots, x_n)$ is equal to zero everywhere in σ_{ij}. Again make use of Lemma 2.1, there is a polynomial $q_2(x_1, \ldots, x_n) \in P_{k-2d}[X]$ such that

$$q_1(x_1, \ldots, x_n) = l_{ij}(x_1, \ldots, x_n) q_2(x_1, \ldots, x_n). \tag{6}$$

Then

$$\eta(x_1, \ldots, x_n) = f|_{\sigma_i}(x_1, \ldots, x_n) - f|_{\sigma_j}(x_1, \ldots, x_n) = [l_{ij}(x_1, \ldots, x_n)]^2 q_2(x_1, \ldots, x_n). \tag{7}$$

Making use of the continuity of the partial derivative of the second order, the third order and up to r order of $f(x_1, \ldots, x_n) \in C^r(\sigma_i \cap \sigma_j)$, we obtain

$$\eta(x_1, \ldots, x_n) = f|_{\sigma_i}(x_1, \ldots, x_n) - f|_{\sigma_j}(x_1, \ldots, x_n) = [l_{ij}(x_1, \ldots, x_n)]^{r+1} q_{ij}(x_1, \ldots, x_n) \tag{8}$$

where $q_{ij}(x_1, \ldots, x_n) \in P_{k-(r+1)d}[X]$. This completes the proof. □

By Theorem 2.2 and Remark 1, we obtain the following theorem immediately.

Theorem 2.3. *Let Δ be a hereditary partition of a n-dimensional simply connected closed semi-algebraic domain Ω in $\mathbb{R}^n$ with mn-cells σ_i. For each adjacent pair σ_i, σ_j of n-cells in Δ, let $\mathbf{V}(l_{ij})$ be the affine hypersurface containing the interior $(n-1)$-cell $\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$, where $l_{ij}(x_1, \ldots, x_n) \in P_{d_{ij}}[X]$ is an irreducible polynomial with total degree d_{ij}. Then*

(1) *In order to have piecewise polynomial function $f \in C_k^r(\Delta)$ existed (indeed piecewise, that is, piecewise polynomial f is not a polynomial function on $\Omega = \bigcup_{i=1}^m \sigma_i$), k and r must satisfy the relation $k \ge (r+1) \cdot \min_{ij} d_{ij}$.*
(2) *A piecewise polynomial function $f \in C_k^r(\Delta)$ if and only if there is a polynomial $q_{ij}(x_1, \ldots, x_n) \in P_{k-(r+1)d_{ij}}[X]$ satisfying the equality (3) for each adjacent pair σ_i, σ_j of n-cells in Δ.*

It is easy to see that $C_k^r(\Delta)$ is a finite-dimensional linear vector space, which called a multivariate spline space with degree k and smoothness r.

In many practical applications such as building cars, airplanes and modelings, blending curves and surfaces, a piecewise polynomial f defined on Δ many not be connected by using the same smoothness degree.

Let $\sigma_{i_1j_1}, \ldots, \sigma_{i_tj_t}$ be a given, fixed, ordering of all interior $(n-1)$-cells in Δ, where $i_s, j_s \in \{1, \ldots, m\}$, and $\sigma_{i_sj_s}$ is common interior boundary between two adjacent n-cells σ_{i_s} and σ_{j_s} for each $s \in \{1, \ldots, t\}$. Let $\mathbf{V}(l_{i_sj_s})$ be the affine hypersurface containing the interior $(n-1)$-cell $\sigma_{i_sj_s} = \sigma_{i_s} \cap \sigma_{j_s}$, where $l_{i_sj_s}(x_1, \ldots, x_n) \in P_{d_{i_sj_s}}[X]$ is an irreducible polynomial with total degree $d_{i_sj_s}$. If a piecewise polynomials $f(x_1, \ldots, x_n)$ of degree k has continuous partial derivatives of order $r_{i_sj_s}$ over $\sigma_{i_sj_s}$, then $f(x_1, \ldots, x_n)$ is called a multivariate spline function with different degree, denoted by $f \in C_k^{\overline{r}}(\Delta)$, where $\overline{r} = (r_{i_1j_1}, \ldots, r_{i_tj_t}) \in \mathbb{Z}_+{}^t$, in which $\mathbb{Z}_+$ denotes the set of nonnegative integers. Obviously $C_k^{\overline{r}}(\Delta) = C_k^r(\Delta)$ when $r = r_{i_1j_1} = \cdots = r_{i_tj_t}$.

From Theorem 2.2 and Remark 1, we can obtain the following theorem immediately.

Theorem 2.4. *Let $\sigma_1, \ldots, \sigma_m$ be all n-cells in Δ and $\sigma_{i_1j_1}, \ldots, \sigma_{i_tj_t}$ all interior $(n-1)$-cells in Δ. Let $f \in P(\Delta)$, and for each i, $1 \le i \le m$, let $f^{[i]} = f|_{\sigma_i} \in \mathbb{R}[x_1, \ldots, x_n]$. Then for any given $\bar{r} = (r_{i_1j_1}, \ldots, r_{i_tj_t}) \in \mathbb{Z_+}^t$, $f \in C_k^{\bar{r}}(\Delta)$ if and only if there is a polynomial $q_{i_sj_s}(x_1, \ldots, x_n) \in P_{k-(r_{i_sj_s}+1)d_{i_sj_s}}[X]$ for each interior $(n-1)$-cell $\sigma_{i_sj_s} = \sigma_{i_s} \cap \sigma_{j_s}$, such that*

$$f^{[i_s]} - f^{[j_s]} = [l_{i_sj_s}(x_1, \ldots, x_n)]^{r_{i_sj_s}+1} q_{i_sj_s}(x_1, \ldots, x_n), \tag{9}$$

where each $l_{i_sj_s}(x_1, \ldots, x_n)$ has the same meaning as above.

The following Theorem 2.5 established on the algebraic hypersurfaces partition Δ generalizes Theorem 2.3 in [4].

Theorem 2.5. *Let $\sigma_1, \ldots, \sigma_m$ be all n-cells in Δ and $\sigma_{i_1j_1}, \ldots, \sigma_{i_tj_t}$ all interior $(n-1)$-cells in Δ, and each $l_{i_sj_s}(x_1, \ldots, x_n)$ has the same meaning as above. Then for any given $\bar{r} = (r_{i_1j_1}, \ldots, r_{i_tj_t})$, a piecewise polynomial f is in $C^{\bar{r}}(\Delta)$ if and only if for each interior $(n-1)$-cell $\sigma_{i_sj_s} = \sigma_{i_s} \cap \sigma_{j_s}$, all coefficients of the polynomial* $\mathrm{rem}(f^{[i_s]} - f^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{i_sj_s})$ *are null, where* rem *is either the usual Euclidean remainder or the pseudo-remainder, and $l_{i_sj_s}(x_1, \ldots, x_n)$ is an irreducible polynomial with degree ≥ 1 with respect to some variable $x_{i_sj_s} \in \{x_1, \ldots, x_n\}$ appearing in $l_{i_sj_s}$. These coefficients are homogeneous linear functions of the coefficients of $f^{[i_s]}$ and $f^{[j_s]}$, and so these nullity conditions may be written into a homogeneous linear system.*

Proof. we can get the conclusion from Theorems 2.2 and 2.4 immediately.

The necessary and sufficient conditions for $f \in C^{\bar{r}}(\Delta)$ in Theorems 2.3–2.5 are called $C^{\bar{r}}(\Delta)$ *smooth continuity condition* of the piecewise polynomial f.

For given $f_\omega \in P(\Delta)$, we know that $f_\omega|_{\sigma_i}$ ($f_\omega^{[i]}$ for short), the restriction of f_ω on cell σ_i, can be written in the form

$$f_\omega|_{\sigma_i} = \sum_{\lambda_1+\cdots+\lambda_n \le k_\omega} a^{[\omega][i]}_{\lambda_1\cdots\lambda_n} x_1^{\lambda_1} \cdots x_n^{\lambda_n} = f_\omega^{[i]}(a^{[\omega][i]}, x_1, x_2, \ldots, x_n), \tag{10}$$

where $\deg(f_\omega) = k_\omega$, and $a^{[\omega][i]}$ is the d_ω-dimensional row vector whose components are formed by coefficients $a^{[\omega][i]}_{\lambda_1\cdots\lambda_n}$ of $f_\omega|_{\sigma_i}$ in fixed order.

Let $Q_\omega = (a^{[\omega][1]}, a^{[\omega][2]}, \ldots, a^{[\omega][m]})^T$. By Theorem 2.5, the homogeneous system of linear algebraic equations with respect to variables Q_ω corresponding to $C^{\bar{r}_\omega}$ smooth continuity condition of the piecewise polynomial f_ω is

$$B_\omega Q_\omega = 0, \tag{11}$$

where the elements in matrix B_ω are determined by coefficients of all polynomials $\mathrm{rem}(f_\omega^{[i_s]} - f_\omega^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{s_{ij}})$ in Theorem 2.5. □

Notation 2. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, we define $X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. Given a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$, we write $\mathrm{cof}(X^\lambda, P)$ for the coefficient of the monomial X^λ in the polynomial P. The monomial X^λ is a monomial of P if $\mathrm{cof}(X^\lambda, P) \ne 0$ and $\mathrm{cof}(X^\lambda, P)X^\lambda$ is a term of P.

The following **BQ Algo** can be used for producing $C^{\bar{r}_\omega}$ smooth continuity condition of the piecewise polynomial f_ω.

Algorithm: BQ Algo

Input: A partition $\Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, of which each element is represented in the form (2); a piecewise polynomial $f_\omega \in P(\Delta)$ with degree k_ω and a t-tuples $\bar{r}_\omega := (r_{i_1j_1}, \ldots, r_{i_tj_t}) \in \mathbb{Z_+}^t$.

Output: $B_\omega Q_\omega = 0$ and Ψ_ω such that Ψ_ω is a $C^{\bar{r}_\omega}$ smooth continuity condition of the piecewise polynomial f_ω.

Step 1 By Theorem 2.5, for each interior $(n-1)$-cell $\sigma_{i_sj_s} = \sigma_{i_s} \cap \sigma_{j_s}$, compute the polynomial $\mathrm{rem}(f_\omega^{[i_s]} - f_\omega^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{i_sj_s})$, where rem, $l_{i_sj_s}$ and $x_{i_sj_s}$ have the same meaning as in Theorem 2.5. Set

$$\Gamma_\omega^{[i_s,j_s]} := \{\mathrm{cof}(X^\lambda, \mathrm{rem}(f_\omega^{[i_s]} - f_\omega^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{i_sj_s})) | X^\lambda \text{ is a monomial of rem } (f_\omega^{[i_s]} - f_\omega^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{s_{ij}})\}.$$

Step 2 When all the $\mathrm{rem}(f_\omega^{[i_s]} - f_\omega^{[j_s]}, l_{i_sj_s}^{r_{i_sj_s}+1}, x_{s_{ij}})$ and the $\Gamma_\omega^{[i_s,j_s]}$ are computed, set $\Gamma_\omega := \bigcup \Gamma_\omega^{[i_s,j_s]}$.

Step 3 Set

$$\Psi_\omega : \bigwedge_{P \in \Gamma_\omega} P = 0, \tag{12}$$

which is $C^{\bar{r}_\omega}$ smooth continuity condition of the piecewise polynomial f_ω by Theorem 2.5, and $\bigwedge_{P\in\Gamma_\omega} P = 0$ is represented as a matrix equation $B_\omega Q_\omega = 0$, then Ψ_ω and $B_\omega Q_\omega = 0$ are what want.

Replacing t-tuples $\bar{r}_\omega$ by a nonnegative integral r_ω in the input above, and set $r_{i_sj_s} := r_\omega$ in step 1, then Ψ_ω in the output is a C^{r_ω} smooth continuity condition of the piecewise polynomial f_ω.

Remark 2. According to algorithm **BQ Algo** and Theorem 2.5, Corollary 2.5 in [4] still holds on the algebraic hypersurfaces partition Δ and can be generalized to the fact that $f_\omega \in C^{\bar{r}_\omega}_{k_\omega}(\Delta)$ if and only if coefficients of polynomials $f_\omega|_{\sigma_1}, \ldots, f_\omega|_{\sigma_m}$ satisfy the system Ψ_ω above (12) or the system $B_\omega Q_\omega = 0$ above (11).

In the input of algorithm **BQ Algo**, we replace the piecewise polynomial f_ω with m-tuple of parametric polynomials $(f_\omega|_{\sigma_1}, \ldots, f_\omega|_{\sigma_m})$, where each $f_\omega|_{\sigma_i}$ is represented in the form (10) and each coefficient $a^{[\omega][i]}_{\lambda_1\cdots\lambda_n}$ of polynomial $f_\omega|_{\sigma_i}$ is regarded as a parameter, and let

$$\overline{\Psi_\omega}: \bigwedge_{P\in\Gamma_\omega} P = 0 \tag{13}$$

be an output of algorithm **BQ Algo**. Then we have the following existence theorem on $C^{\bar{r}_\omega}$ piecewise polynomial.

Theorem 2.6. (1) *$\overline{\Psi_\omega}$ above is a system of homogeneous linear equations with respect to variables $Q_\omega = (a^{[\omega][1]}, a^{[\omega][2]}, \ldots, a^{[\omega][m]})^T$.*

(2) *$\dim(\overline{\Psi_\omega})$ denoting the dimension of solutions space of the system $\overline{\Psi_\omega}$. Piecewise polynomial $f_\omega \in C^{\bar{r}_\omega}_{k_\omega}(\Delta)$ exists, if and only if $\dim(\overline{\Psi_\omega}) > \binom{k_\omega+n}{n}$.*

(3) *The dimension of multivariate spline space $C^{\bar{r}_\omega}_{k_\omega}(\Delta)$ is equal to $\dim(\overline{\Psi_\omega})$.*

Proof. We can get the conclusion (1) from algorithm **BQ Algo** and Theorems 2.3–2.5 immediately.

Since the Δ is a hereditary partition and the dimension of polynomial space $P_{k_\omega}[X]$ with total degree $\leq k_\omega$ equals to $\binom{k_\omega+n}{n}$, it follows from algorithm **BQ Algo** and Theorems 2.3–2.5 that piecewise polynomial $f_\omega \in C^{\bar{r}_\omega}_{k_\omega}(\Delta)$ exists (indeed piecewise) if and only if $\dim(\overline{\Psi_\omega}) > \binom{k_\omega+n}{n}$. Moreover, $\dim(C^{\bar{r}_\omega}_{k_\omega}(\Delta)) = \dim(\overline{\Psi_\omega})$. This completes the proof. □

For convenience, we suppose piecewise polynomial $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$ in the rest sections. In fact, the results and methods in Sections 3–6 are still efficient for piecewise polynomials $f_\omega \in C^{\bar{r}_\omega}_{k_\omega}(\Delta)$. □

3. Parametric piecewise polynomial system

In this section, We show that solving a parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ amounts to solve m parametric semi-algebraic systems $\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}$ with $U = (u_1, u_2, \ldots, u_d)$ as parameters, and also present an algorithm to produce $\{\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}\}$ and $\Pi_d(\Theta_{\mathrm{BQ}})$.

Let $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$, $\omega = 1, \ldots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{R}[V][X]$ for each n-cell σ_i in Δ. Define a parametric piecewise polynomial system (PPS for short) as follows

$$\mathrm{PPS}: \begin{cases} f_1(V,X) = 0, \\ f_2(V,X) = 0, \\ \cdots \\ f_n(V,X) = 0, \\ \bigwedge_{P\in\Gamma_\omega} P = 0, \quad \omega = 1, \ldots, n \end{cases} \tag{14}$$

where $V = (u_1, u_2, \ldots, u_\tau)$ and $X = (x_1, x_2, \ldots, x_n)$ are viewed as parameters and variables, respectively; $\bigwedge_{P\in\Gamma_\omega} P = 0$ has the same meaning as (12) in the algorithm **BQ Algo**.

We know from algorithm **BQ Algo**, Theorem 2.5, system (11) and Corollary 2.5 in [4] that the following system BQ is a system of polynomial equations with respect to variables $V = (u_1, u_2, \ldots, u_\tau)$

$$\mathrm{BQ}: \begin{cases} \bigwedge_{P\in\Gamma_1} P = 0, \\ \bigwedge_{P\in\Gamma_2} P = 0, \\ \cdots \\ \bigwedge_{P\in\Gamma_n} P = 0 \end{cases} \tag{15}$$

and $\bigwedge_{P\in\Gamma_\omega} P = 0$ in BQ is a necessary and sufficient condition of $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$ for each $\omega \in \{1, \ldots, n\}$.

Based on the fact above and by a similar argument to that in the proof of Theorem 2.5 in [4] we have

Theorem 3.1. *Let $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$, $\omega = 1, \ldots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{R}[V][X]$ for each n-cell σ_i in Δ. Then the* PPS *system above has the same the real zero set as $\mathcal{Z}(f_1, \ldots, f_n)$ for any given values of the parameters V's.*

Let $\Theta_{\mathrm{BQ}} = \{(u_1, u_2, \ldots, u_\tau) \in \mathbb{R}^\tau \mid \bigwedge_{P\in\Gamma_\omega} P = 0, \omega = 1, \ldots, n\}$ be the set of real solutions of the system BQ.

Let $d \le \tau$. Define a mapping $\Pi_d : \mathbb{R}^\tau \longrightarrow \mathbb{R}^d$ by

$$\Pi_d(u_1, \ldots, u_d, u_{d+1}, \ldots, u_\tau) := (u_1, u_2, \ldots, u_d), \qquad (u_1, \ldots, u_d, u_{d+1}, \ldots, u_\tau) \in \mathbb{R}^\tau. \tag{16}$$

Remark 3. Usually there are so many parameters in the system PPS and equations in the system BQ that direct solve them is complicated (see the example in Section 6), so we need to reduce the system BQ to its simple form G_{BQ}, and use the G_{BQ} to eliminate as many parameters in $f_1(V, X), \ldots, f_n(V, X)$ as possible and to reduce the system PPS, and the set of real solutions of the system PPS also remains unchanged.

We propose the following algorithm **SimPPS** to handle above problems.

Algorithm: SimPPS

Input: A partition $\Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, of which each element is represented in the form (2); parametric piecewise polynomials $f_1, \ldots, f_n$ with $f_1|_{\sigma_i}, \ldots, f_n|_{\sigma_i} \in \mathbb{R}[V][X]$ (for $i = 1, \ldots, m$); nonnegative integral $r_1, \ldots, r_n$.

Output: G_{BQ} and $\{\varphi_1(U, X), \ldots, \varphi_n(U, X)\}$, where G_{BQ} is a set of polynomials with variables V and $\varphi_1(U, X), \ldots, \varphi_n(U, X)$ are parametric piecewise polynomials on Δ with $U = (u_1, u_2, \ldots, u_d)$ as parameters and $\varphi_\omega|_{\sigma_i} \in \mathbb{R}[U][X]$ for each ω, i, such that

• G_{BQ} is a reduced set of the system BQ, and the set Θ_{BQ} is the same as the set $\Theta_{G_{\mathrm{BQ}}} = \{(u_1, u_2, \ldots, u_\tau) \in \mathbb{R}^\tau \mid \bigwedge_{P \in G_{\mathrm{BQ}}} P = 0\}$;

• $\varphi_1(U, X), \ldots, \varphi_n(U, X)$ are reduced parametric piecewise polynomials of $f_1, \ldots, f_n$, and for any given parameter values $V \in \Theta_{\mathrm{BQ}}$, the system PPS above (14) has the same the set of real solutions as the following parametric piecewise polynomial system

$$\mathrm{RPPS}: \begin{cases} \varphi_1(U, X) = 0, \\ \cdots \\ \varphi_n(U, X) = 0, \\ \bigwedge\limits_{P \in G_{\mathrm{BQ}}} P = 0, \end{cases} \tag{17}$$

where $U = \Pi_d(V)$.

Step 1 For each piecewise polynomial $f_\omega, \omega = 1, \ldots, n$, use algorithm **BQ Algo** to obtain Γ_ω and the system Ψ_ω : $\bigwedge_{P \in \Gamma_\omega} P = 0$, and so get the system BQ. Let the parametric PPS as above (14).

Step 2 Compute G_{RBQ} the reduced Gröbner basis of $\bigcup_{\omega=1}^{n} \Gamma_\omega$ w.r.t. $>_V$ where $>_V$ is the lex order with $u_1 > u_2 > \cdots > u_\tau$. Let $G_{\mathrm{RBQ}} = \{g_1, g_2, \ldots, g_k\}$.

Step 3 For each polynomial $g_i \in G_{\mathrm{RBQ}}$, removing factors which are positive (or negative) identically in $\mathbb{R}^\tau$ and multiple factors from g_i, we get polynomial h_i. When all g_i in G_{RBQ} are handled, set $G_{\mathrm{SBQ}} = \{h_1, h_2, \ldots, h_k\}$.

Step 4 Compute G_{BQ} the reduced Gröbner basis of G_{SBQ} w.r.t. $>_V$.

Step 5 For each $\omega \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m\}$, compute $\overline{f_\omega|_{\sigma_i}(V, X)}^{G_{\mathrm{BQ}}}$ the remainder on division of $f_\omega|_{\sigma_i}(V, X)$ by G_{BQ}. Without loss of generality, suppose that $\tau - d$ parameters $Y = (u_{d+1}, u_{d+2}, \ldots, u_\tau)$ in $f_1(V, X), \ldots, f_n(V, X)$ are eliminated by this method.

Step 6 Define parametric piecewise polynomials on Δ, $\varphi_1(U, X), \varphi_2(U, X), \ldots, \varphi_n(U, X)$, such that $\varphi_\omega|_{\sigma_i}(U, X) = \overline{f_\omega|_{\sigma_i}(V, X)}^{G_{\mathrm{BQ}}}$ for each $\omega \in \{1, \ldots, n\}$ and σ_i in Δ.

Step 7 Output G_{BQ} and $\{\varphi_1(U, X), \ldots, \varphi_n(U, X)\}$, which are what we want.

Proof of correctness: $\Theta_{\mathrm{BQ}} = \Theta_{G_{\mathrm{BQ}}}$ by steps 1–4. This together with step 5 and step 6 implies that for any given parameter values $V \in \Theta_{\mathrm{BQ}}$, the system PPS has the same the set of real solutions as the system RPPS.

Define a parametric piecewise polynomial system as follows

$$\overline{\mathrm{PPS}}: \begin{cases} \varphi_1(U, X) = 0, \\ \varphi_2(U, X) = 0, \\ \cdots \\ \varphi_n(U, X) = 0 \end{cases} \tag{18}$$

where $\varphi_1(U, X), \ldots, \varphi_n(U, X) \in P(\Delta)$, with $\varphi_\omega|_{\sigma_i} \in \mathbb{R}[U][X]$ for each ω, i, have the same meaning as (17) in the algorithm **SimPPS**, and $U = (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}})$ (the methods of computing $\Pi_d(\Theta_{\mathrm{BQ}})$ can be found in [2,20,21]) are viewed as parameters.

Define:

$$\mathrm{SAS}\overline{\mathrm{S}}_i : \begin{cases} \varphi_1|_{\sigma_i}(U, X) = 0, \ldots, \varphi_n|_{\sigma_i}(U, X) = 0, \\ g_1^{[i]}(X) > 0, \ldots, g_{\gamma_i}^{[i]}(X) > 0. \end{cases} \tag{19}$$

for $i = 1, 2, \ldots, m$, where $\varphi_1, \ldots, \varphi_n$ and $g_1^{[i]}(X), \ldots, g_{\gamma_i}^{[i]}(X)$ have the same meaning as (18) and (2), respectively; $U = (u_1, u_2, \ldots, u_d)$ and $X = (x_1, x_2, \ldots, x_n)$ are viewed as parameters and variables, respectively.

Let $TRzero(\cdot)$ and $Rzero(\cdot)$ denote the set of torsion-free real zeros and the set of real zeros of a given system, respectively, and $TRzero(\cdot)|_{\sigma_i}$ the set $TRzero(\cdot) \bigcap \sigma_i$.

The following Theorem 3.2 shows that Theorem 2.6 in [4] still holds on the new algebraic hypersurfaces partition Δ. However the proof methods of two Theorems are different.

Theorem 3.2. *Let Δ be a hereditary partition of an n-dimensional simply connected closed semi-algebraic domain Ω in $\mathbb{R}^n$ with mn-cells σ_i. Let $f_\omega \in C^{r_\omega}_{k_\omega}(\Delta)$, $\omega = 1, \dots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{R}[V][X]$ for each n-cell σ_i in Δ. Then for any given $U \in \mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}})$ and $V \in \Theta_{\mathrm{BQ}}$ with $U = \Pi_d(V)$, the piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_n)$, the PPS, the $\overline{\mathrm{PPS}}$ and these $\mathrm{SAS}\overline{S_i}$ have the following relations:*

(1)

$$TRzero(\mathcal{Z}(f_1, \dots, f_n)) = TRzero(\mathrm{PPS}) = TRzero(\overline{\mathrm{PPS}}) = \bigcup_{i=1}^{m} Rzero(\mathrm{SAS}\overline{S_i})$$

and

$$TRzero(\mathcal{Z}(f_1, \dots, f_n))|_{\sigma_i} = TRzero(\overline{\mathrm{PPS}})|_{\sigma_i} = Rzero(\mathrm{SAS}\overline{S_i})$$

for all $i = 1, 2, \dots, m$.

(2) *if the parametric $\mathcal{Z}(f_1, \dots, f_n)$ is zero-dimensional, then for any given $i \in \{1, 2, \dots, m\}$,*

$$NTRzero(\mathcal{Z}(f_1, \dots, f_n))|_{\sigma_i} = NTRzero(\overline{\mathrm{PPS}})|_{\sigma_i} = NRzero(\mathrm{SAS}\overline{S_i}),$$

where $NTRzero()|_{\sigma_i}$ and $NRzero(\cdot)$ mean the number of distinct torsion-free real zeros in cell σ_i and the number of distinct real zeros of a given system, respectively.

Proof. According to the algorithm **SimPPS**, $\Theta_{\mathrm{BQ}} = \Theta_{G_{\mathrm{BQ}}} = \{(u_1, u_2, \dots, u_\tau) \in \mathbb{R}^\tau \mid \bigwedge_{P \in G_{\mathrm{BQ}}} P = 0\}$, this implies that for any given parameter values $U \in \Pi_d(\Theta_{\mathrm{BQ}})$ and $V \in \Theta_{\mathrm{BQ}}$ with $U = \Pi_d(V)$, the system $\overline{\mathrm{PPS}}$ has the same the set of real solutions as the system RPPS. On the other hand, the system PPS has the same the set of real solutions as the system RPPS for any given parameter values $V \in \Theta_{\mathrm{BQ}}$, Hence we have the conclusion immediately by (19) and Theorem 3.1. This completes the proof. □

Hence, we only need to consider the parametric $\overline{\mathrm{PPS}}$ w.r.t. the parameters U's values when characterize the real zeros of the parametric $\mathcal{Z}(f_1, \dots, f_n)$(number of real zeros, existence of a parameterization, etc.) w.r.t. the parameters V's values. Theorem 3.2 also shows that solving the parametric piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_n)$ amounts to solve m parametric semi-algebraic systems $\mathrm{SAS}\overline{S_1}, \dots, \mathrm{SAS}\overline{S_m}$ with $U = (u_1, u_2, \dots, u_d)$ as parameters.

Remark 4. Although solving the parametric piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_n)$ amounts to solve m parametric semi-algebraic systems $\mathrm{SAS}\overline{S_1}, \dots, \mathrm{SAS}\overline{S_m}$, but we should solve these parametric semi-algebraic systems as a whole, instead of solve them separately, since the parameters U's values are the same in parametric semi-algebraic systems $\mathrm{SAS}\overline{S_i}\mathrm{SAS}\overline{S_1}, \dots, \mathrm{SAS}\overline{S_m}$.

4. Solving parametric piecewise polynomial system

In this section, we show that solving parametric piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_n)$ is reduced to the computation of m discriminant varieties $W_{\mathrm{SAS}\overline{S_i}}$ of $\mathrm{SAS}\overline{S_i}$, $i = 1, \dots, m$. The variety $\bigcup_{i=1}^m W_{\mathrm{SAS}\overline{S_i}}$ can then be used to solve the $\mathcal{Z}(f_1, \dots, f_n)$ as long as one can describe $\mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}}) \setminus (\bigcup_{i=1}^m W_{\mathrm{SAS}\overline{S_i}})$.

Lazard and Rouillier (see [14]) recently proposed a new framework for studying the basic constructible set

$$\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) \neq 0, \dots, f_l(x) \neq 0\}$$

and the basic semi-algebraic set

$$\mathcal{S} = \{x \in \mathbb{R}^n,\ p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) > 0, \dots, f_l(x) > 0\},$$

where $p_i, f_i \in \mathbb{Q}[U, X]$, $x = [U, X]$, $U = [u_1, \dots, u_d]$ is the set of parameters and $X = [x_{d+1}, \dots, x_n]$ the set of unknowns.

Let $\prod_U$ denote the canonical projection onto parameter's space and $\overline{\mathcal{C}}$ the Zariski closure of the constructible subset $\mathcal{C}$ of $\mathbb{C}^n$. The Zariski closure of $\mathcal{C}$ coincides with its closure for the usual topology of $\mathbb{C}^n$. Let δ be the dimension of $\overline{\prod_U(\mathcal{C})}$. An algebraic variety W is a discriminant variety of $\mathcal{C}$ w.r.t. $\prod_U$ if and only if:

- W is contained in $\overline{\prod_U(\mathcal{C})}$ and $W = \overline{\prod_U(\mathcal{C})}$ if and only if $\prod_U^{-1}(u) \cap \mathcal{C}$ is infinite for almost all $u \in \overline{\prod_U(\mathcal{C})}$.
- The connected components $\mathcal{U}_1, \dots, \mathcal{U}_k$ of $\overline{\prod_U(\mathcal{C})} \setminus W$ are analytic submanifolds of dimension δ, and $(\prod_U^{-1}(\mathcal{U}_i) \cap \mathcal{C}, \prod_U)$ is an analytic covering of $\mathcal{U}_i$ for $i = 1, \dots, k$. (This guarantees that the cardinality of $\prod_U^{-1}(u) \cap \mathcal{C}$ is constant on neighborhood of u, that $\prod_U^{-1}(\mathcal{U}_i) \cap \mathcal{C}$ is a finite collection of sheets and that $\prod_U$ is a local diffeomorphism from each of these sheets onto $\mathcal{U}$).

Lazard and Rouillier obtained the following result in [14] and proposed an algorithm to compute the discriminant variety W of $\mathcal{C}$ w.r.t. $\prod_U$ and the minimal discriminant variety W_D of $\mathcal{C}$ w.r.t. $\prod_U$.

Proposition 4.1 (*See [14, Proposition 6]*). *Let $\mathcal{S}$ and $\mathcal{C}$ be defined as above and $W \subset \mathbb{C}^n$ be a discriminant variety of $\mathcal{C}$ w.r.t. $\prod_U$. If $W \neq \overline{\prod_U(\mathcal{C})}$, then $(\overline{\prod_U(\mathcal{C})} \setminus W) \cap \mathbb{R}^d$ has a finite number of connected components which are real analytic manifolds; if $\mathcal{U}$ is such a component, the number of points of $\mathcal{S}$ over $\mathcal{U}$ is constant and $(\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}, \prod_U)$ is a real analytic covering of $\mathcal{S}$ if $\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}$ is not empty.*

We define

$$\mathcal{C}_{\mathrm{SAS}\overline{S_i}} := \{x \in \mathbb{C}^{d+n}, \varphi_1|_{\sigma_i}(U,X) = 0, \ldots, \varphi_n|_{\sigma_i}(U,X) = 0, g_1^{[i]}(X) \neq 0, \ldots, g_{\gamma_i}^{[i]}(X) \neq 0\} \tag{20}$$

and

$$\mathcal{S}_{\mathrm{SAS}\overline{S_i}} := \{x \in \mathbb{R}^{d+n}, \varphi_1|_{\sigma_i}(U,X) = 0, \ldots, \varphi_n|_{\sigma_i}(U,X) = 0, g_1^{[i]}(X) > 0, \ldots, g_{\gamma_i}^{[i]}(X) > 0\} \tag{21}$$

for $i = 1, 2, \ldots, m$, where $\mathrm{SAS}\overline{S_i}$, $\varphi_1|_{\sigma_i}(U,X), \ldots, \varphi_n|_{\sigma_i}(U,X)$ and $g_1^{[i]}(X), \ldots, g_{\gamma_i}^{[i]}(X)$ have the same meaning as (19) and $x = (U, X)$, where $U = [u_1, \ldots, u_d]$ is the set of parameters and $X = [x_1, \ldots, x_n]$ the set of unknowns.

To describe our results more precisely, in the remainder of the paper, $\prod_U : \mathbb{C}^{d+n} \longrightarrow \mathbb{C}^d$, defined by $\prod_U(u_1, \ldots, u_d, x_1, \ldots, x_n) = (u_1, \ldots, u_d)$, denote the canonical projection onto parameter's space.

Definition 4.2. Let $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta)$, $\omega = 1, \ldots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{Q}[V][X]$ for each n-cell σ_i in Δ. Let $W_{\mathrm{SAS}\overline{S_i}} \subset \mathbb{C}^d$ be a discriminant variety of $\mathcal{C}_{\mathrm{SAS}\overline{S_i}}$ w.r.t. $\prod_U$ for $i = 1, 2, \ldots, m$. $W_{\mathcal{Z}} := \bigcup_{i=1}^m W_{\mathrm{SAS}\overline{S_i}}$ is called a discriminant variety of the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ w.r.t. $\prod_U$.

The $W_{\mathcal{Z}}$ is also called a discriminant variety of the parametric piecewise polynomial system $\overline{\mathrm{PPS}}$ w.r.t. $\prod_U$.

Theorem 4.3. *Let $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta)$, $\omega = 1, \ldots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{Q}[V][X]$ for each n-cell σ_i in Δ. Let $W_{\mathcal{Z}} \subset \mathbb{C}^d$ be a discriminant variety of the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ w.r.t. $\prod_U$. If the parametric $\mathcal{Z}(f_1, \ldots, f_n)$ be zero-dimensional, then we have:*

(i) $(\mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}})) \setminus W_{\mathcal{Z}}$ *has a finite number of connected components which are real analytic manifolds; if $\mathcal{U}$ is such a component, the number of points of $\mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ over $\mathcal{U}$ is constant and $(\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}_{\mathrm{SAS}\overline{S_i}}, \prod_U)$ is a real analytic covering of $\mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ if $\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ is not empty;*

(ii) *The number of torsion-free points in σ_i of $\mathcal{Z}(f_1, \ldots, f_n)$ over $\mathcal{U}$ is constant for each n-cells σ_i in Δ. Further the number of torsion-free points of $\mathcal{Z}(f_1, \ldots, f_n)$ over $\mathcal{U}$ is also constant, and $(\prod_U^{-1}(\mathcal{U}) \cap TRzero(\mathcal{Z}(f_1, \ldots, f_n)), \prod_U)$ is a real analytic covering of $TRzero(\mathcal{Z}(f_1, \ldots, f_n))$ if $\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ is not empty for $i = 1, 2, \ldots, m$.*

Proof. It is clear from [14] that the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ is zero-dimensional, which implies that $W_{\mathrm{SAS}\overline{S_i}} \neq \overline{\prod_U(\mathcal{C}_{\mathrm{SAS}\overline{S_i}})}$ and $\overline{\prod_U(\mathcal{C}_{\mathrm{SAS}\overline{S_i}})} = \mathbb{C}^d$ for $i = 1, 2, \ldots, m$. Thus we have that

$$\bigcap_{i=1}^{m} \left(\overline{\prod_U (\mathcal{C}_{\mathrm{SAS}\overline{S_i}})} \setminus W_{\mathrm{SAS}\overline{S_i}} \right) = \mathbb{C}^d \setminus \bigcup_{i=1}^{m} W_{\mathrm{SAS}\overline{S_i}} = \mathbb{C}^d \setminus W_{\mathcal{Z}},$$

and so $\mathbb{R}^d \setminus W_{\mathcal{Z}}$ has a finite number of connected components by Proposition 4.1.

It is easy to see from [1,2] that $\Pi_d(\Theta_{\mathrm{BQ}})$ is a semi-algebraic subsets of $\mathbb{R}^d$, and so $\Pi_d(\Theta_{\mathrm{BQ}})$ is the disjoint union of a finite number of connected semi-algebraic subsets of $\mathbb{R}^d$. This together with Proposition 4.1 implies that $(\mathbb{R}^d \cap \Pi_d(\Theta_{\mathrm{BQ}})) \setminus W_{\mathcal{Z}}$ has a finite number of connected components which are real analytic manifolds, and if $\mathcal{U}$ is such a component, the number of points of $\mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ over $\mathcal{U}$ is constant and $(\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}_{\mathrm{SAS}\overline{S_i}}, \prod_U)$ is a real analytic covering of $\mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ if $\prod_U^{-1}(\mathcal{U}) \cap \mathcal{S}_{\mathrm{SAS}\overline{S_i}}$ is not empty. This shows that Property (i) holds.

Property (ii) follows easily from (i) and Theorem 3.2. This completes the proof. □

Theorem 4.3 shows that solving $\mathcal{Z}(f_1, \ldots, f_n)$ is reduced to the computation of a discriminant variety $W_{\mathcal{Z}}$ of the parametric $\mathcal{Z}(f_1, \ldots, f_n)$ w.r.t. $\prod_U$. The following **DISCRVAR OF PPAV** can be used for producing $W_{\mathcal{Z}}$.

Algorithm: DISCRVAR OF PPAV

Input: A partition $\Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, of which each element is represented in the form (2); parametric piecewise polynomials $f_1, \ldots, f_n$ with $f_1|_{\sigma_i}, \ldots, f_n|_{\sigma_i} \in \mathbb{Q}[V][X]$ (for $i = 1, \ldots, m$); nonnegative integral $r_1, \ldots, r_n$.

Output: $\Pi_d(\Theta_{\mathrm{BQ}})$, $W_{\mathcal{Z}}$, $\{\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}\}$ and the parametric system $\overline{\mathrm{PPS}}$.

Step 1 Perform SimPPS. Set G_{BQ} and $\{\varphi_1(U,X), \ldots, \varphi_n(U,X)\}$ to output.

Step 2 Use the methods described in [2] or Wang's methods [20,21] to compute $\Pi_d(\Theta_{G_{BQ}})$ (i.e. $\Pi_d(\Theta_{BQ})$). Let the parametric system $\overline{\mathrm{PPS}}$ as (18) and $\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}$ as (19).

Step 3 Set $W_{\mathcal{Z}} := \emptyset$, $i := 1$.

Step 4 Use the algorithm provided in [14] and implemented in Moroz's DV package or Liang's Rootfinding[Parametric] package (see [22]) to compute the discriminant variety $W_{\mathrm{SAS}\overline{S_i}}$ of $\mathcal{C}_{\mathrm{SAS}\overline{S_i}}$ w.r.t. $\prod_U$. Set $W_{\mathcal{Z}} := W_{\mathcal{Z}} \bigcup W_{\mathrm{SAS}\overline{S_i}}$, $i := i + 1$.

Step 5 If $i \leq m$, go to step 3; Else, output $\Pi_d(\Theta_{G_{BQ}})$, $W_{\mathcal{Z}}$, $\{\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}\}$ and $\overline{\mathrm{PPS}}$, which are what we want.

Theorem 4.3 also shows that The variety $W_{\mathcal{Z}}$ can then be used to solve the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ as long as one can describe $(\mathbb{R}^d \cap \Pi_d(\Theta_{BQ})) \setminus W_{\mathcal{Z}}$. This can be done by using the critical points method [15] and the Collins PCAD [11–13] or an *open* cylindrical algebraic decomposition [18,16,17].

Remark 5. In order to make our algorithm efficient in practice, we consider that the solutions of the parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ are defined inside the n-cells but not on the intersections of their boundaries (i.e. their faces).

We can deal with the situation that the solutions of the $\mathcal{Z}(f_1, \ldots, f_n)$ are defined on the faces (say $\sigma_{ij} = \sigma_i \cap \sigma_j \in \Delta$, it is the intersection of two n-cell σ_i and σ_j, and is contained in the affine hypersurface $\mathbf{V}(g_1^{[i]}(X))$, where $g_1^{[i]}(X)$ has the same meaning as (2)), by adding the following parametric piecewise polynomial system (22) into the set $\{\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}\}$, and adding the discriminant variety $W_{F_{ij}}$ of $F_{ij}\overline{S_i}$ w.r.t. $\prod_U$ into $W_{\mathcal{Z}}$. According to Theorem 4.3, the variety $W_{\mathcal{Z}} \bigcup W_{F_{ij}}$ and $\{\mathrm{SAS}\overline{S_1}, \ldots, \mathrm{SAS}\overline{S_m}, F_{ij}\overline{S_i}\}$ can be used to solve the parametric system $\mathcal{Z}(f_1, \ldots, f_n)$ of which solutions lie inside the n-cells or on the face σ_{ij}.

$$F_{ij}\overline{S_i} : \begin{cases} \varphi_1|_{\sigma_i}(U, X) = 0, \ldots, \varphi_n|_{\sigma_i}(U, X) = 0, \\ g_1^{[i]}(X) = 0, \\ g_2^{[i]}(X) > 0, \ldots, g_{\gamma_i}^{[i]}(X) > 0. \end{cases} \tag{22}$$

where $\varphi_1, \ldots, \varphi_n$ and $g_1^{[i]}(X), \ldots, g_{\gamma_i}^{[i]}(X)$ have the same meaning as (19).

Remark 6. Actually, we do not consider either that the solutions of the parametric system $\mathcal{Z}(f_1, \ldots, f_n)$ are defined on the "boundaries" of the face σ_{ij} (i.e. faces of σ_{ij}). We can deal with the situation that the solutions of the $\mathcal{Z}(f_1, \ldots, f_n)$ are defined on the "boundaries" by a similar approach in Remark 4.

5. Classification method

Let $\mathcal{Z}(f_1, \ldots, f_n)$ be a zero-dimensional parametric piecewise polynomial system, where $f_\omega \in C_{k_\omega}^{r_\omega}(\Delta)$, $\omega = 1, \ldots, n$, and $f_\omega|_{\sigma_i} \in \mathbb{Q}[V][X]$ for each n-cell σ_i in Δ; $V = (u_1, u_2, \ldots, u_\tau)$ is the set of parameters and $X = (x_1, x_2, \ldots, x_n)$ the set of variables. We want to be able to answer to the following question:

"Given a parametric piecewise polynomial system $\mathcal{Z}(f_1, \ldots, f_n)$ and integers $N_1, \ldots, N_m$, does there exist an open set $\mathcal{O}$ in the parameters' space Θ_{BQ} such that for all $p_0 \in \mathcal{O}$, the zero-dimensional non-parametric piecewise polynomial system $\mathcal{Z}_{p_0}(f_1, \ldots, f_n)$ obtained by specializing at point p_0 have exactly $N_1, \ldots, N_m$ distinct torsion-free real zeros in n-cells $\sigma_1, \ldots, \sigma_m$ respectively? If yes, give explicitly a point $a \in \mathcal{O}$".

As discussed at the ending of Section 3, above problem is equivalent to the question that for the parametric system $\overline{\mathrm{PPS}}$ as (18), does there exist an open set $\mathcal{O}$ in the parameters' space $\Pi_d(\Theta_{BQ})$ such that for all $p_0 \in \mathcal{O}$, the non-parametric system $\overline{\mathrm{PPS}}_{p_0}$ obtained by specializing at point p_0 have exactly $N_1, \ldots, N_m$ distinct torsion-free real zeros in n-cells $\sigma_1, \ldots, \sigma_m$ respectively? if yes, give explicitly a point $a \in \mathcal{O}$.

For this purpose, we present a method, which is similar to that in [15], to classify the parametric values p_0 of a dense open set of $\Pi_d(\Theta_{BQ})$ according to the number of distinct torsion-free real zeros of $\overline{\mathrm{PPS}}_{p_0}$ in every n-cell in Δ. The method we describe in this article computes exactly an open classification of $\Pi_d(\Theta_{BQ})$ with relation to $\overline{\mathrm{PPS}}$ according to the following definition:

Definition 5.1 (*Open Classification*)**.** Let $\mathcal{Z}(f_1, \ldots, f_n)$ be a zero-dimensional parametric piecewise polynomial system and the parametric system $\overline{\mathrm{PPS}}$ as (18). Let $\mathcal{J}$ be a finite subset of $\mathbb{Z}_+^m$ ($\mathbb{Z}_+$ denotes the set of nonnegative integers) and $\mathcal{O}_{[N_1,\ldots,N_m]}$, $[N_1, \ldots, N_m] \in \mathcal{J}$, be open sets (for the euclidean topology) in the parameters' space $\Pi(\Theta_{BQ})$ such that:

- $\forall p_0 \in \mathcal{O}_{[N_1,\ldots,N_m]}$, $\overline{\mathrm{PPS}}_{p_0}$ (or $\mathcal{Z}_{p_0}(f_1, \ldots, f_n)$) have exactly $N_1, \ldots, N_m$ distinct torsion-free real zeros in n-cells $\sigma_1, \ldots, \sigma_m$ respectively;
- $\bigcup_{[N_1,\ldots,N_m]\in\mathcal{J}} \mathcal{O}_{[N_1,\ldots,N_m]}$ is dense in parameters' space $\Pi(\Theta_{BQ})$.

We call the *family* $\{\mathcal{O}_{[N_1,\ldots,N_m]}\}_{[N_1,\ldots,N_m]\in\mathcal{J}}$ an open classification of $\Pi(\Theta_{BQ})$ with relation to $\overline{\mathrm{PPS}}$ (or $\mathcal{Z}(f_1, \ldots, f_n)$).

Remark 7. In particular, the complementary in $\Pi_d(\Theta_{\mathrm{BQ}})$ of a discriminant variety of $\overline{\mathrm{PPS}}$ defines an open classification of $\Pi_d(\Theta_{\mathrm{BQ}})$ with relation to $\overline{\mathrm{PPS}}$.

Computing an open classification. Given a parametric piecewise polynomial system $\mathcal{Z}(f_1, \dots, f_n)$, we show that an open classification of $\Pi_d(\Theta_{\mathrm{BQ}})$ with relation to $\overline{\mathrm{PPS}}$ can be represented by (q, F, ϕ), which are defined as follows:

- q is a polynomial and a discriminant variety of $\overline{\mathrm{PPS}}$;
- F is a set of rational points in each connected component of $q \neq 0$ and $F \subset \Pi_d(\Theta_{\mathrm{BQ}})$;
- ϕ is a table which associates to each point p_0 of F the number of distinct torsion-free real zeros of $\overline{\mathrm{PPS}}_{p_0}$ in every n-cell in Δ.

In this representation, each $\mathcal{O}_{[N_1,\dots,N_m]}$ is represented by q and the subset of points $\phi^{-1}([N_1, \dots, N_m]) \in F$ such that:

$\mathcal{O}_{[N_1,\dots,N_m]} = \{x \in \Pi_d(\Theta_{\mathrm{BQ}}) |$ there exists $p \in \phi^{-1}([N_1, \dots, N_m])$ and a continuous path in $\Pi_d(\Theta_{\mathrm{BQ}})$ from p to x included in $q \neq 0\}$

The following **Algo-Clas-Rep OF PPAV** can be used for producing a representation of an open classification.

Algorithm: Algo-Clas-Rep OF PPAV

Input: A partition $\Delta = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$, of which each element is represented in the form (2); parametric piecewise polynomials $f_1, \dots, f_n$ with $f_1|_{\sigma_i}, \dots, f_n|_{\sigma_i} \in \mathbb{Q}[V][X]$ (for $i = 1, \dots, m$); nonnegative integral $r_1, \dots, r_n$.

Output: the 3-tuple (q, F, ϕ).

Step 1 Perform the DISCRVAR OF PPAV. Set $\Pi_d(\Theta_{G_{\mathrm{BQ}}})$, $W_{\mathcal{Z}}$, $\{\mathrm{SAS}\overline{S_1}, \dots, \mathrm{SAS}\overline{S_m}\}$ and the parametric system $\overline{\mathrm{PPS}}$ to output.

Step 2 The discriminant variety q. For the first step, q as a polynomial vanishing at the discriminant variety $W_{\mathcal{Z}}$ of $\overline{\mathrm{PPS}}$.

Step 3 The sampling points F. The critical point method allows to compute at least one point in each connected component of a semi-algebraic set defined by strict inequalities. Algorithms using these methods is given in [2,15–17]. In this step, the function *sampling_inequations* in Safey's RAGlib (Release 2.32) package is used to compute the sampling points F in $\Pi_d(\Theta_{\mathrm{BQ}})$, which is a finite set of point in each connected component, which has a non-empty intersection with $\Pi_d(\Theta_{\mathrm{BQ}})$, of the semi-algebraic set defined by $q \neq 0$. In this step, we only consider the case where $\Pi_d(\Theta_{\mathrm{BQ}})$ is defined by strict inequalities and do not consider the situation on the "boundaries" of $\Pi_d(\Theta_{\mathrm{BQ}})$.

Step 4 The table ϕ. Finally, we compute a table where each point p_0 of F is associated to the number of distinct torsion-free real zeros of $\overline{\mathrm{PPS}}_{p_0}$ in each n-cell in Δ. For this step, using *realzeros* command in DISCOVERER package developed by Bican Xia (see [10,23,24]).

Remark 8. In this step 2, we omit the situation on the "boundaries" of $\Pi_d(\Theta_{\mathrm{BQ}})$. Actually, in many cases, the 3-tuple (q, F, ϕ) obtained by this algorithm is satisfactory enough because we do not lose too much information. We can deal with the situation when parameter U are on the "boundaries" of $\Pi_d(\Theta_{\mathrm{BQ}})$ (say $h(U) = 0, h_1(U) > 0, \dots, h_k(U) > 0$), by using the critical point method to sample points on an algebraic set (see [2,15–17]), on the set: $h(U) = 0$ and $T \cdot q(U)h_1(U) \cdots h_k(U) - 1 = 0$, that is, the set $Q = h^2 + (Tqh_1 \cdots h_k - 1)^2 = 0$, where T is a new variable, then according to the signs of $h_1(U), \dots, h_k(U)$ at these sampling points, we can obtain sampling points F by doing projection $\prod_U$.

6. Examples

In this section we present some experimental results of our algorithms.

We denote by Δ_+ the partition of $\mathbb{R}^2$ consisting of two orthogonal lines. Without loss of generality, assume that two partition lines are x and y axes, respectively. We mark the third quadrant σ_1 (as source cell) and the others cells of Δ_+, $\sigma_2, \sigma_3, \sigma_4$, in a counter-clockwise manner.

Let $f_1(x, y), f_2(x, y) \in P(\Delta_+)$, where

$$f_1(x,y) = \begin{cases} f_1(x,y)|_{\sigma_1} = x^3 + u_1xy + y^2 + u_2, \\ f_1(x,y)|_{\sigma_2} = (u_6u_4^3 - 1)y^3 + (u_2 - u_3)x^2y + u_6y^2 + u_7^2 - 10, \\ f_1(x,y)|_{\sigma_3} = u_{10}x^3 + (u_{11}^2 - 4)x^2y + u_{12}y^3 + (3 - u_6u_8)y^2 + u_{13}x + u_{14}, \\ f_1(x,y)|_{\sigma_4} = u_8x^3 + (2u_4^2 - 2)y^3 + (2u_1^2 + 2u_6)y^2 + u_9, \end{cases} \tag{23}$$

$$f_2(x,y) = \begin{cases} f_2(x,y)|_{\sigma_1} = u_3x^2 + (3u_1 + u_9)xy + 5u_{13}x + u_{15}, \\ f_2(x,y)|_{\sigma_2} = u_{13}u_9x^2 + u_{16}xy + (u_5 + 2u_3)y + 1, \\ f_2(x,y)|_{\sigma_3} = u_{21}x^2 + u_{22}xy + (2 - 3u_8)y^2 + u_{23}, \\ f_2(x,y)|_{\sigma_4} = u_{17}x^2 + u_{18}xy + (u_5u_{21} - 1)y^2 + (u_{19}^2 - u_3 - 1)x + u_{20}. \end{cases} \tag{24}$$

Use the algorithm **SimPPS**, we get that $f_1(x, y) \in C_3^1(\Delta_+)$ and $f_2(x, y) \in C_2^1(\Delta_+)$ if and only if these parameters coefficient of $f_1(x, y)$ and $f_2(x, y)$ satisfy the following system

$$G_{\mathrm{BQ}} : \begin{cases} u_1 = u_{10} = u_{12} = u_{13} = u_{21} = 0, \\ u_4 = u_6 = u_8 = u_{15} = u_{20} = u_{23} = 1, \\ u_{16} = u_{18} = u_{22} = u_9 = u_{14} = u_2 = u_7^2 - 10, \\ u_5 = -2u_3, \qquad u_{17} = u_3, \\ u_{11}^2 - 4 = u_2 - u_3, \qquad u_{19}^2 = u_3 + 1. \end{cases} \tag{25}$$

Use the algorithm DISCRVAR OF PPAV, we have

$$\mathrm{SAS}\overline{S_1} : \begin{cases} \varphi_1(x, y)|_{\sigma_1} = x^3 + y^2 + u_2 = 0, \\ \varphi_2(x, y)|_{\sigma_1} = u_3x^2 + u_2xy + 1 = 0, \\ -x > 0, \qquad -y > 0, \end{cases} \tag{26}$$

$$\mathrm{SAS}\overline{S_2} : \begin{cases} \varphi_1(x, y)|_{\sigma_2} = (u_2 - u_3)x^2y + y^2 + u_2 = 0, \\ \varphi_2(x, y)|_{\sigma_2} = u_2xy + 1 = 0, \\ x > 0, \qquad -y > 0, \end{cases} \tag{27}$$

$$\mathrm{SAS}\overline{S_3} : \begin{cases} \varphi_1(x, y)|_{\sigma_3} = (u_2 - u_3)x^2y + 2y^2 + u_2 = 0, \\ \varphi_2(x, y)|_{\sigma_3} = u_2xy - y^2 + 1 = 0, \\ x > 0, \qquad y > 0, \end{cases} \tag{28}$$

$$\mathrm{SAS}\overline{S_4} : \begin{cases} \varphi_1(x, y)|_{\sigma_4} = x^3 + 2y^2 + u_2 = 0, \\ \varphi_2(x, y)|_{\sigma_4} = u_3x^2 + u_2xy - y^2 + 1 = 0, \\ -x > 0, \qquad y > 0. \end{cases} \tag{29}$$

and

$$\Pi_d(\Theta_{\mathrm{BQ}}) = \{(u_2, u_3) \in \mathbb{R}^2 | u_2 \geq -10, u_3 \geq -1, -u_3 + u_2 + 4 \geq 0\} \tag{30}$$

and

$$\begin{aligned} W_{\mathcal{Z}} := \{& u_2, u_2 + 2, u_2 - u_3, 4u_2^7 + 27u_2^2 - 54u_2u_3 + 27u_3^2, u_2^2u_3^3 + 1, 108u_2^{15} + 16u_2^8u_3^6 + 1080u_2^{12}u_3 + 128u_2^5u_3^7 \\ & + 3420u_2^9u_3^2 + 256u_2^2u_3^8 + 3240u_2^6u_3^3 - 160u_2^3u_3^4 + 256u_3^5 + 3125u_2^4, 72u_2^9 + 256u_2^2u_3 - 4096u_2^2u_3^3 \\ & + 288u_2^5u_3^2 - 1024u_2^3u_3^3 + 6144u_2^3u_3^2 - 4096u_2^4u_3 + 1968u_2^4u_3^2 + 1024u_3^4 - 1888u_2^5u_3 - 16u_2^7u_3 - 576u_2^6u_3 \\ & - 4096u_2u_3^3 + 1024u_2u_3^4 - 4096u_2^3u_3 + 1024u_2^5 + 27u_2^8u_3^2 + 8u_2^8 + 1024u_2^4 + 688u_2^6 + 32u_2^{11} - 54u_2^9u_3 \\ & + 72u_2^7u_3^2 + 6144u_2^2u_3^2 - 144u_2^8u_3 + 8u_2^6u_3^2 + 27u_2^{10} + 288u_2^7, 725760u_2^9u_3^4 + 1327104u_2^3u_3^6 + 4055040u_2^7u_3^4 \\ & + 1302912u_2^10u_3^3 + 109440u_2^{11}u_3^2 + 8748u_2^{10}u_3 + 49152u_2^6u_3^8 + 3110400u_2^8u_3^4 + 331776u_2^7u_3 - 16384u_2^3u_3^5 \\ & + 110592u_2^5u_3^6 + 131072u_2^5u_3^8 + 65536u_2^4u_3^9 + 1539648u_2^6u_3 + 1327104u_2u_3^3 + 65536u_2^4u_3^8 + 34992u_2^8u_3^2 \\ & + 262144u_2^3u_3^9 + 746496 + 52704u_2^9 + 17280u_2^14u_3 + 46656u_2^6u_3^3 + 399168u_2^11u_3^3 + 65536u_2^2u_3 \\ & + 111456u_2^13u_3^2 + 1410048u_2^2u_3^3 + 912384u_2^5u_3^2 + 1119744u_2^4u_3^3 + 1603584u_2^3u_3^3 + 1024u_2^10u_3^6 \\ & + 12288u_2^8u_3^7 + 4672512u_2^3u_3^2 + 1440000u_2^4u_3 + 2661120u_2^4u_3^2 + 15552u_2^{15}u_3 + 1492992u_2 + 1119744u_2^2 \\ & + 226800u_2^8 + 729u_2^{12} + 46656u_2^4 + 373248u_2^3 + 884736u_3^3 + 200000u_2^6 + 2916u_2^{11} + 2764800u_2^2u_3 \\ & + 245376u_2^{12}u_3^3 + 864u_2^{17} + 1149696u_2^9u_3^3 + 262144u_2u_3^6 + 46656u_2^9u_3 + 2777088u_2^4u_3^5 + 207360u_2^8u_3^3 \\ & + 5160960u_2^2u_3^5 + 1468416u_2^6u_3^4 + 233280u_2^7u_3^2 - 20480u_2^5u_3^4 + 66096u_2^8u_3 + 950272u_2^2u_3^6 + 373248u_2^5u_3^3 \\ & + 528768u_2^6u_3^2 + 3041280u_2^6u_3^5 + 972u_2^{10} + 16384u_2^7u_3^7 + 580608u_2^7u_3^5 + 360000u_2^7 + 663552u_2^4u_3^6 \\ & + 262144u_2^2u_3^9 + 262144u_2^10 + 2534400u_2^5u_3\}. \end{aligned}$$

Use the algorithm Algo-Clas-Rep OF PPAV, the polynomial q is a product of all polynomials in $W_{\mathcal{Z}}$, and we get 312 sampling points, the mean time, of which the command *realzeros* computes the number of distinct torsion-free real zeros of each corresponding zero-dimensional $\overline{\mathrm{PPS}}_{p_0}$ on each 2-cell in Δ, is about 0.2 s.

Finally, we can recover the fact that the parametric $\mathcal{Z}(f_1, f_2)$ may have exactly [1, 0, 0, 1], [1, 0, 0, 2], [1, 0, 1, 1], [1, 0, 2, 1], [0, 0, 1, 1] or [0, 1, 0, 1] torsion-free real zeros. We present in Table 1 a sample point in the parameters' space where the $\mathcal{Z}(f_1, f_2)$ has $[N_1, N_2, N_3, N_4]$ torsion-free real zeros.

Table 1
Sample parametric points corresponding to a wanted number of torsion-free real zeros.

Number of zeros in $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ respectively	u_2	u_3
[1, 0, 0, 1]	$-\frac{6031465858}{1286863861}$	$\frac{2170874275}{2573727722}$
[1, 0, 0, 2]	$-\frac{2446986389}{1036457590}$	$-\frac{5172463427}{6614398502}$
[1, 0, 1, 1]	$-\frac{1810934727}{954567023}$	$\frac{2276000566}{1189893549}$
[1, 0, 2, 1]	$-\frac{5011264745}{5369028568}$	$-\frac{10136510801}{10485872495}$
[0, 0, 1, 1]	$\frac{5995561738}{3569399071}$	$\frac{1441416751}{2529238715}$
[0, 1, 0, 1]	$\frac{136311515333}{20993894}$	$\frac{235490593321}{72537771}$

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11071031, U0935004, 10271022, 60373093, 60533060 and 11026086), the Zhejiang Provincial Natural Science Foundation (No. Y7080068) and the Foundation of Department of Education of Zhejiang Province (No. 20070628).

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